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Compound-Commuting Mappings on Skew-Hermitian Matrices

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Abstract

Let $\mathbb{F}$ be a field with proper involution $-$ and let r, s be even integers with $r, s > 2$. Let $\mathcal{SH}_r(\mathbb{F})$ and $C_{r-1}(M)$ denote the set of all $r \times r$ skew-Hermitian matrices over the field $\mathbb{F}$ and the $(r-1)$-th compound of a matrix M, respectively. In this study, we investigate the characterization of a mapping $\zeta\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_s(\mathbb{F})$ that satisfies,

$$\zeta(C_{r-1}(M + \gamma N)) = C_{s-1}(\zeta(M) + \gamma\zeta(N)),$$

for any $M, N \in \mathcal{SH}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$, where $\mathbb{F}^- = \{x \in \mathbb{F} \mid \overline{x} = x\}$.

Keywords: preserver problems; compound-commuting mappings; skew-Hermitian matrices; rank.

1 Introduction

Let $\mathbb{M}_r(\mathbb{F})$ denote the set of all $r \times r$ matrices over the field $\mathbb{F}$. Let M be a matrix in $\mathbb{M}_r(\mathbb{F})$. We use M_{ij}, M^T, $|M|$, $R(M)$, $M[i \mid j]$, M^ρ, and $M^\sim$ to represent the (i,j)-entry of M, the transpose of M, the determinant of M, the rank of M, the matrix obtained by eliminating the i-th row and j-th column from M, the matrix obtained from M by applying ρ entrywise, and the matrix whose (i,j)-th entry is $M_{r+1-j,r+1-i}$, respectively. We also use I_r, 0_r, and E_{ij} to represent the identity matrix in $\mathbb{M}_r(\mathbb{F})$, the zero matrix in $\mathbb{M}_r(\mathbb{F})$, and the matrix whose (i,j)-th entry is 1 and other entries are 0, respectively. We let $J_r = \sum_{i=1}^{r} E_{i,r+1-i}$ and $Z_r = \sum_{i=1}^{r} (-1)^{i+1} E_{ii}$. It can be checked that $M^\sim = J_r M J_r$. To avoid confusion, we write $||S||$ to denote the cardinality of a set S.

Let $\mathbb{F}$ be a field. A bijective mapping $-: \mathbb{F} \to \mathbb{F}$ is an involution of $\mathbb{F}$ if $\overline{\overline{x}} = x$ (i.e., applying the mapping $-$ twice returns the original element) for any $x \in \mathbb{F}$, $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{xy} = \overline{y}\,\overline{x}$ for any $x, y \in \mathbb{F}$. We define $\mathbb{F}^- = \{x \in \mathbb{F} \mid \overline{x} = x\}$ and $S\mathbb{F}^- = \{x \in \mathbb{F} \mid \overline{x} = -x\}$. Note that if $\mathbb{F} = \mathbb{F}^-$, then $-$ is the identity involution of $\mathbb{F}$, otherwise, $-$ is a proper involution of $\mathbb{F}$. We denote by $\mathcal{SH}_r(\mathbb{F})$ (respectively, $\mathcal{H}_r(\mathbb{F})$) the set of all $r \times r$ skew-Hermitian (respectively, Hermitian) matrices over the field $\mathbb{F}$. If $M = -\overline{M}^T$ (respectively, $M = \overline{M}^T$), then $M \in \mathcal{SH}_r(\mathbb{F})$ (respectively, $M \in \mathcal{H}_r(\mathbb{F})$), where $-$ is applied on M entrywise. The $(r-1)$-th compound of a matrix M, denoted by $C_{r-1}(M)$, has the (i,j)-th entry of $C_{r-1}(M)$ is defined by,

$$(C_{r-1}(M))_{ij} = |M[r+1-i \mid r+1-j]|.$$

In the following, we present some basic properties of $C_{r-1}(M)$ as stated by Chooi [6].

Lemma 1.1. ([6, *Lemmas 2.2 and 2.3*]) *Let $\mathbb{F}$ be a field with involution $-$ and let $r \in \mathbb{N}$ with $r > 1$. Let $M, N \in \mathbb{M}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}$. Then the following assertions are valid.*

(a) $C_{r-1}(0_r) = 0_r$.

(b) $C_{r-1}(I_r) = I_r$.

(c) $C_{r-1}(\gamma M) = \gamma^{r-1} C_{r-1}(M)$.

(d) $C_{r-1}(MN) = C_{r-1}(M) C_{r-1}(N)$.

(e) $C_{r-1}(M^{-1}) = C_{r-1}(M)^{-1}$ *when the rank of M is r.*

(f) $C_{r-1}(M^T) = C_{r-1}(M)^T$.

(g) $C_{r-1}(\overline{M}) = \overline{C_{r-1}(M)}$.

(h) $C_{r-1}(M^\sim) = C_{r-1}(M)^\sim$.

(i) $C_{r-1}(M^\rho) = C_{r-1}(M)^\rho$, *where ρ is a field homomorphism of $\mathbb{F}$.*

(j) *If $M = J_r$ or $M = Z_r$, then* $C_{r-1}(M) = \begin{cases} -M, & \textit{if} \quad r \equiv 0, 3 \pmod 4, \\ M, & \textit{otherwise.} \end{cases}$

(k) $R(C_{r-1}(M)) = r$, *if and only if* $R(M) = r$.

(l) $R(C_{r-1}(M)) = 1$, *if and only if* $R(M) = r - 1$.

(m) $R(C_{r-1}(M)) = 0$, *if and only if* $R(M) \leqslant r - 2$.

(n) $C_{r-1}(M) = Z_r \operatorname{adj}(M)^{\sim} Z_r$, *where* $\operatorname{adj}(M)$ *stands for the adjoint matrix of* M.

(o) $C_{r-1}(I_r - E_{r+1-i,r+1-i} + (\gamma - 1)E_{jj}) = \gamma E_{ii}$ *for any* $i, j \in \mathbb{N}$ *with* $1 \leqslant i \leqslant r$ *and* $1 \leqslant j \neq r + 1 - i \leqslant r$.

(p) $C_{r-1}(I_r - E_{r+1-i,r+1-i} - E_{r+1-j,r+1-j} + (-1)^{i+j+1}\gamma E_{r+1-j,r+1-i}) = \gamma E_{ij}$ *for any* $i, j \in \mathbb{N}$ *with* $1 \leqslant i \neq j \leqslant r$.

Matrix theory is important in both mathematical research and practical applications across various fields. Bayram and Kaplan [1] shows its application in computer algebra whereas Chern and Teh [4] used Parikh matrix in understanding structural properties of words and their numerical properties. The investigation of invariants plays a significant role in matrix theory. One well-known area within matrix theory is Linear Preserver Problems. Linear Preserver Problems is considered an advanced topic in matrix theory because it involve the characterization of linear mappings that preserve certain invariants between matrix spaces. The earliest research on Linear Preserver Problems can probably be traced back to the paper [11]. Frobenius [11] demonstrated that a linear mapping $f \colon \mathbb{M}_r(\mathbb{C}) \to \mathbb{M}_r(\mathbb{C})$ that preserves the function (i.e., $|f(M)| = |M|$) for all $M \in \mathbb{M}_r(\mathbb{C})$ is of the form,

$$f(M) = PMQ \text{ for any } M \in \mathbb{M}_r(\mathbb{C}),$$

or

$$f(M) = PM^TQ \text{ for any } M \in \mathbb{M}_r(\mathbb{C}),$$

where $P, Q \in \mathbb{M}_r(\mathbb{C})$ with $|PQ| = 1$. For the motivation of Linear Preserver Problems, refer to the survey paper [13]. Li and Tsing [14] described some techniques used in Linear preserver problems. [15] surveyed on not only linear but also additive preserver problems.

Let matrix spaces M_1 and M_2 over the same field, if $M \in M_i$ for $i \in \{1, 2\}$, then $\Upsilon(M) \in M_i$ where Υ is a matrix function. A mapping $\psi \colon M_1 \to M_2$ that satisfies $\psi(\Upsilon(M)) = \Upsilon(\psi(M))$ for any $M \in M_1$ is called a Υ-commuting mapping. Linear mappings commuting with certain transformations is also a type of Linear Preserver Problems. Sinkhorn [19] was a pionner to discussing this kind of problem by considering $\Upsilon(M) = \operatorname{adj}(M)$. He studied adjoint-commuting linear mapping on $\mathbb{M}_r(\mathbb{C})$. One the basis of the classical theorem of Frobenius [11] concerning determinant preservers, Sinkhorn used the continuity argument and proved that for $r \geqslant 3$, the mapping $\psi \colon \mathbb{M}_r(\mathbb{C}) \to \mathbb{M}_r(\mathbb{C})$ is of the form,

$$\psi(M) = \mu PMP^{-1} \text{ for any } M \in \mathbb{M}_r(\mathbb{C}),$$

or,

$$\psi(M) = \mu PM^TP^{-1} \text{ for any } M \in \mathbb{M}_r(\mathbb{C}),$$

where $\mu \in \mathbb{C}$ with $\mu^{r-2} = 1$ and $P \in \mathbb{M}_r(\mathbb{C})$ with P is invertible. In [3], Chan et al. generalized Sinkhorn's result from the complex field to arbitrary infinite fields. In the same paper, the authors considered $\Upsilon(M) = e^M$. They determined the structures of linear mappings on square and symmetric matrices that commute with the exponential function. In [2], Chan and Lim were interested in $\Upsilon(M) = M^k$ for certain fixed integers $k > 1$. They characterized linear mappings that commute with the k-th power function on square matrices.

With the in-depth study of Linear Preserver Problems, the authors in [12, 16] proposed replacing the linearity assumption with additivity or homogeneity assumptions. Later on, some researchers studied Υ-commuting mappings on various matrix spaces by dropping the linearity assumption. For example, the adjoint-commuting additive mappings on block triangular matrices were investigated by Chooi [5], additive rank-1 preservers on Hermitian matrices were studied by Tang [21].

Let $r, s \in \mathbb{N}$ with $r, s > 2$. Let M_1 and M_2 be matrix spaces over the same field. A mapping $\zeta\colon M_1 \to M_2$ is said to be a compound-commuting mapping if ζ satisfies,

$$\zeta(C_{r-1}(M)) = C_{s-1}(\zeta(M)) \text{ for any } M \in M_1.$$

The study of compound-commuting mappings on various matrix spaces was initiated by Chooi [6]. In recent years, many researchers do not consider strong assumptions like linearity, additivity, or homogeneity on Preserver Problems. For instance, Dolinar and Šemrl [9] studied determinant preserver problems by imposing weaker assumptions. They showed that a surjective mapping $f\colon \mathbb{M}_r(\mathbb{C}) \to \mathbb{M}_r(\mathbb{C})$ satisfying,

$$|M + \gamma N| = |f(M)| + \gamma|f(N)| \text{ for any } M, N \in \mathbb{M}_r(\mathbb{C}) \text{ and } \gamma \in \mathbb{C},$$

is linear without imposing the linearity assumption. In other words, they showed that Frobenius's result in [11] still holds true when the linearity assumption is removed and replaced with weaker assumptions. Tan and Wang [20] further improved the work of Dolinar and Šemrl [9]. They removed the surjectivity assumption and demonstrated that Frobenius's result holds true for any field with more than r elements. Chooi and Ng [7, 8] investigated the general form of adjoint-commuting mappings on various matrix spaces without imposing strong assumptions.

Motivated by their works, this study focuses on compound-commuting mappings on skew-Hermitian matrices under weaker conditions. Our aim is to investigate the characterization of a mapping $\zeta\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_s(\mathbb{F})$ that satisfies,

$$\zeta(C_{r-1}(M + \gamma N)) = C_{s-1}(\zeta(M) + \gamma\zeta(N)), \qquad \text{[S1]}$$

for any $M, N \in \mathcal{SH}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$.

Before we proceed to the next section, we provide some examples of compound-commuting mapping on skew-Hermitian matrices that satisfy [S1].

Example 1.1. *Let $\mathbb{F}$ be a field with proper involution $-$ and let r, s be even integers with $r, s > 2$.*

(a) *Let $k \in \mathbb{N}$ with $k \leqslant s - 2$. Let $\zeta_1\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_s(\mathbb{F})$ be defined by,*

$$\zeta_1(M) = \begin{cases} \sum_{i=1}^{k} M_{ii}E_{ii}, & \textit{if} \quad 1<R(M)<r, \\ 0, & \textit{otherwise.} \end{cases}$$

(b) *Let $k \in \mathbb{N}$ with $2 \leqslant k \leqslant s$. Let $\zeta_2\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_s(\mathbb{F})$ be defined by,*

$$\zeta_2(M) = \begin{cases} \sum_{i=2}^{k} (M_{1i}E_{1i} + M_{i1}E_{i1}), & \textit{if} \quad 1<R(M)<r, \\ 0, & \textit{otherwise.} \end{cases}$$

(c) *Let $k \in \mathbb{N}$ and $s > r + k$. Let $f\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_r(\mathbb{F})$ be a non-zero mapping and define $\zeta_3\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_s(\mathbb{F})$ by,*

$$\zeta_3(M) = \begin{cases} 0_2 \oplus f(M) \oplus 0_{s-r-2}, & \textit{if} \quad 1<R(M)<r, \\ 0, & \textit{otherwise.} \end{cases}$$

By applying Lemma 1.1(k)-(m), we observe that each ζ_i satisfies,

$$\zeta_i(C_{r-1}(M+\gamma N)) = 0_s = C_{s-1}(\zeta_i(M)+\gamma\zeta_i(N)),$$

for any $M, N \in \mathcal{SH}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$. Additionally, we see that each ζ_i is neither injective nor surjective, and it is not necessary for r and s to be the same. This shows that each ζ_i does not necessarily have a general form. In this study, we aim to determine the general form of $\zeta\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_s(\mathbb{F})$ that satisfies [S1].

2 Preliminaries of the Main Result

First, we establish some relationships between Hermitian matrices and skew-Hermitian matrices.

Lemma 2.1. *Let $\mathbb{F}$ be a field with involution $-$ and let $r \in \mathbb{N}$. Let $M \in \mathbb{M}_r(\mathbb{F})$ and $\mu \in S\mathbb{F}^-$ with $\mu \neq 0$. Then, the following assertions are valid.*

(a) *$M \in \mathcal{H}_r(\mathbb{F})$ if and only if $\mu M \in \mathcal{SH}_r(\mathbb{F})$.*

(b) *$M \in \mathcal{SH}_r(\mathbb{F})$ if and only if $\mu M \in \mathcal{H}_r(\mathbb{F})$.*

Proof.

(a) If $M \in \mathcal{H}_r(\mathbb{F})$, then $\overline{(\mu M)}^T = (\overline{\mu})(\overline{M}^T) = -\mu M$. This leads to $\mu M \in \mathcal{SH}_r(\mathbb{F})$. Conversely, since $\mu M \in \mathcal{SH}_r(\mathbb{F})$, it follows that $-\mu M = \overline{(\mu M)}^T = (\overline{\mu})(\overline{M}^T) = -\mu \overline{M}^T$. This yields that $M = \overline{M}^T$ and consequently, $M \in \mathcal{H}_r(\mathbb{F})$.

(b) If $M \in \mathcal{SH}_r(\mathbb{F})$, then $\overline{(\mu M)}^T = (\overline{\mu})(\overline{M}^T) = (-\mu)(-M) = \mu M$. This leads to $\mu M \in \mathcal{H}_r(\mathbb{F})$. Conversely, since $\mu M \in \mathcal{H}_r(\mathbb{F})$, it follows that $\mu M = \overline{(\mu M)}^T = (\overline{\mu})(\overline{M}^T) = -\mu \overline{M}^T$. This yields that $M = -\overline{M}^T$ and consequently, $M \in \mathcal{SH}_r(\mathbb{F})$.

□

Lemma 2.2. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let μ be a fixed arbitrary non-zero element in $S\mathbb{F}^-$ and let ζ be a mapping from $\mathcal{SH}_r(\mathbb{F})$ to $\mathcal{SH}_s(\mathbb{F})$. If $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ is a mapping satisfying,*

$$h(W) = \mu^{-1}\zeta(\mu W) \text{ for any } W \in \mathcal{H}_r(\mathbb{F}),$$

then, the following equations are equivalent.

(a) *$h(\mu^{r-2}C_{r-1}(M+\gamma N)) = \mu^{s-2}C_{s-1}(h(M)+\gamma h(N))$ for any $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$.*

(b) *$\zeta(C_{r-1}(M+\gamma N)) = C_{s-1}(\zeta(M)+\gamma\zeta(N))$ for any $M, N \in \mathcal{SH}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$.*

Proof. (a) $\Rightarrow$ (b): For any $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$, we have,

$$\begin{aligned}\zeta(C_{r-1}(M+\gamma N)) &= \mu h(\mu^{-1}C_{r-1}(M+\gamma N))\\ &= \mu h(\mu^{r-2}\mu^{-(r-1)}C_{r-1}(M+\gamma N))\\ &= \mu h(\mu^{r-2}C_{r-1}(\mu^{-1}M+\gamma\mu^{-1}N))\\ &= \mu\mu^{s-2}C_{s-1}(h(\mu^{-1}M)+\gamma h(\mu^{-1}N))\\ &= \mu^{s-1}C_{s-1}(h(\mu^{-1}M)+\gamma h(\mu^{-1}N))\\ &= \mu^{s-1}C_{s-1}(\mu^{-1}\zeta(M)+\gamma\mu^{-1}\zeta(N))\\ &= \mu^{s-1}\mu^{-(s-1)}C_{s-1}(\zeta(M)+\gamma\zeta(N))\\ &= C_{s-1}(\zeta(M)+\gamma\zeta(N)).\end{aligned}$$

(b) $\Rightarrow$ (a): For any $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$, we have,

$$\begin{aligned}h(\mu^{r-2}C_{r-1}(M+\gamma N)) &= \mu^{-1}\zeta(\mu^{r-1}C_{r-1}(M+\gamma N))\\ &= \mu^{-1}\zeta(C_{r-1}(\mu M+\gamma\mu N))\\ &= \mu^{-1}C_{s-1}(\zeta(\mu M)+\gamma\zeta(\mu N))\\ &= \mu^{-1}C_{s-1}(\mu h(M)+\gamma\mu h(N))\\ &= \mu^{-1}\mu^{s-1}C_{s-1}(h(M)+\gamma h(N))\\ &= \mu^{s-2}C_{s-1}(h(M)+\gamma h(N)).\end{aligned}$$

$\square$

Note that, if r is an even positive integer, then $\mu^{r-2} \in \mathbb{F}^-$ for any $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$. In the following, we investigate the properties of $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ that satisfies,

$$h(\mu^{r-2}C_{r-1}(M+\gamma N)) = \mu^{s-2}C_{s-1}(h(M)+\gamma h(N)), \qquad [C1]$$

for any $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$.

These properties are essential in proving the main results.

Lemma 2.3. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. For $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$, then the following assertions are valid.*

(a) $h(0_r) = 0_s$.

(b) $h(\mu^{r-2}C_{r-1}(M)) = \mu^{s-2}C_{s-1}(h(M))$.

(c) $C_{s-1}(h(M+\gamma N)) = C_{s-1}(h(M)+\gamma h(N))$.

Proof.

(a) $$\begin{aligned}h(0_r) &= h(\mu^{r-2}0_r) = h(\mu^{r-2}C_{r-1}(0_r)) = h(\mu^{r-2}C_{r-1}(0_r+(-1)0_r))\\ &= \mu^{s-2}C_{s-1}(h(0_r)+(-1)h(0_r)) = \mu^{s-2}C_{s-1}(h(0_r)-h(0_r))\\ &= \mu^{s-2}C_{s-1}(0_s) = \mu^{s-2}0_s\\ &= 0_s.\end{aligned}$$

(b) For any $M \in \mathcal{H}_r(\mathbb{F})$, we have,

$$\begin{aligned} h(\mu^{r-2}C_{r-1}(M)) &= h(\mu^{r-2}C_{r-1}(M+0_r)) = h(\mu^{r-2}C_{r-1}(M+0M)) \\ &= \mu^{s-2}C_{s-1}(h(M)+0h(M)) = \mu^{s-2}C_{s-1}(h(M)+0_s) \\ &= \mu^{s-2}C_{s-1}(h(M)). \end{aligned}$$

(c) By part (b), we obtain,

$$\mu^{s-2}C_{s-1}(h(M+\gamma N)) = h(\mu^{r-2}C_{r-1}(M+\gamma N)) = \mu^{s-2}C_{s-1}(h(M)+\gamma h(N)).$$

Consequently, we have,

$$C_{s-1}(h(M+\gamma N)) = C_{s-1}(h(M)+\gamma h(N)).$$

□

Next, we state some fascinating results regarding the rank of a Hermitian matrix.

Lemma 2.4. [6, *Lemma 2.4*] *Let $\mathbb{F}$ be a field with involution $-$ and let $r \in \mathbb{N}$ with $r > 1$. If $M \in \mathcal{H}_r(\mathbb{F})$ with $R(M) = 1$, then $M = C_{r-1}(W)$ for some $W \in \mathcal{H}_r(\mathbb{F})$ with $R(W) = r-1$.*

Lemma 2.5. [8, *Lemma 2.4*] *Let $\mathbb{F}$ be a field with involution $-$ and let $r \in \mathbb{N}$ with $r > 2$. Let $M, N \in \mathcal{H}_r(\mathbb{F})$. Then, the following assertions are valid.*

(a) *If $R(M) = t$, then $R(M+W) = r$ for some $W \in \mathcal{H}_r(\mathbb{F})$ with $R(W) = r-t$.*

(b) *$R(M+W) = R(N+W) = r$ for some $W \in \mathcal{H}_r(\mathbb{F})$.*

(c) *$R(M+W) = r$ for some non-zero matrix $W \in \mathcal{H}_r(\mathbb{F})$ with either $R(M) = r$ or $R(W) = r$ but not both.*

(d) *If $M \neq 0_r$, then $R(M+W) = r-1$ for some $W \in \mathcal{H}_r(\mathbb{F})$ with $R(W) \leqslant r-2$.*

(e) *If $||\mathbb{F}^-|| \geqslant r+2$ and $R(M+N) = r$, then $R(M+N+\mu N) = r$ for some $\mu \in \mathbb{F}^-$ with $\mu \neq 0$.*

Here, we determine some relations between $R(M)$ and $R(h(M))$.

Lemma 2.6. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. Let $M \in \mathcal{H}_r(\mathbb{F})$. Then, the following assertions are valid.*

(a) *$R(h(M)) \leqslant 1$ if $R(M) = 1$.*

(b) *$R(h(M)) \leqslant s-1$ if $R(M) = r-1$.*

(c) *$R(h(M) \leqslant s-2$ if $R(M) \leqslant r-2$.*

(d) *$R(M) = r$ if $R(h(M)) = s$.*

Proof.

(a) If $R(M) = 1$, then we obtain $R(C_{r-1}(M)) = 0$ by Lemma 1.1(m). Thus,

$$\mu^{s-2}C_{s-1}(h(M)) = h(\mu^{r-2}C_{r-1}(M)) = h(\mu^{r-2}0_r) = h(0_r) = 0_s.$$

This leads to,
$$C_{s-1}(h(M)) = 0_s.$$
This shows that $R(C_{s-1}(h(M))) = 0$. Hence, by Lemma 1.1(m), $R(h(M)) \leqslant s-2$. On the other hand, since $R(\mu^{2-r}M) = R(M) = 1$, it follows from Lemma 2.4 that $\mu^{2-r}M = C_{r-1}(N)$ for some $N \in \mathcal{H}_r(\mathbb{F})$ with $R(N) = r-1$. This implies that $M = \mu^{r-2}C_{r-1}(N)$. So, we obtain,
$$h(M) = h(\mu^{r-2}C_{r-1}(N)) = \mu^{s-2}C_{s-1}(h(N)).$$
It follows that $R(\mu^{s-2}C_{s-1}(h(N))) \leqslant s-2$. Therefore, by Lemma 1.1(k)-(m), we obtain,
$$R(\mu^{s-2}C_{s-1}(h(N))) = R(C_{s-1}(h(N))) \leqslant 1.$$
Consequently, we obtain $R(h(M)) \leqslant 1$.

(b) If $R(M) = r-1$, then we obtain $R(C_{r-1}(M)) = 1$ by Lemma 1.1(l). Since,
$$R(\mu^{r-2}C_{r-1}(M)) = R(C_{r-1}(M)) = 1,$$
we obtain $R(h(\mu^{r-2}C_{r-1}(M))) \leqslant 1$ by part (a). By Lemma 1.1(m), it follows that,
$$R(C_{s-1}(h(\mu^{r-2}C_{r-1}(M)))) = 0,$$
and so, $C_{s-1}(h(\mu^{r-2}C_{r-1}(M))) = 0_s$. On the other hand, we have,
$$C_{s-1}(h(\mu^{r-2}C_{r-1}(M))) = C_{s-1}(\mu^{s-2}C_{s-1}(h(M))) = (\mu^{s-2})^{s-1}C_{s-1}(C_{s-1}(h(M))).$$
This implies that,
$$C_{s-1}(C_{s-1}(h(M))) = 0_s.$$
This means that $R(C_{s-1}(C_{s-1}(h(M))) = 0$. So, by Lemma 1.1(k)-(m), we obtain,
$$R(C_{s-1}(h(M))) \leqslant s-2,$$
and consequently, $R(h(M)) \leqslant s-1$.

(c) If $R(M) \leqslant r-2$, then we obtain $R(C_{r-1}(M)) = 0$ by Lemma 1.1(m). This implies that,
$$\mu^{s-2}C_{s-1}(h(M)) = h(\mu^{r-2}C_{r-1}(M)) = h(\mu^{r-2}0_r) = h(0_r) = 0_s.$$
It follows that,
$$C_{s-1}(h(M)) = 0_s.$$
Hence, $R(C_{s-1}(h(M))) = 0$ and consequently, we obtain $R(h(M)) \leqslant s-2$ by Lemma 1.1(m).

(d) Since $R(h(M)) = s$, it follows from part (b) that $R(M) \neq r-1$. If $R(M) \leqslant r-2$, then by part (c), we obtain $R(h(M)) \leqslant s-2$, which is a contradiction. Consequently, we conclude that $R(M) = r$.

□

Next, we are interested in the properties of h when it is injective.

Lemma 2.7. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. Let $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$. Then, the following assertions are equivalent.*

(a) h *is injective.*

(b) $\ker(h) = \{0_r\}$.

(c) $R(M) = r$ *if and only if* $R(h(M)) = s$.

(d) $R(M + \gamma N) = r$ *if and only if* $R(h(M) + \gamma h(N)) = s$.

Proof. (a) $\Rightarrow$ (b): Let $M \in \ker(h)$. Hence, we have $h(M) = 0_s = h(0_r)$. Since h is injective, we have $M = 0_r$. Consequently, we obtain $\ker(h) = \{0_r\}$.

(b) $\Rightarrow$ (c): The sufficiency part follows from Lemma 2.6(d) immediately. For necessity part, we suppose to the contrary that $R(h(M)) < s$. Note that,

$$\begin{aligned} h(\mu^{r-2}C_{r-1}(\mu^{r-2}C_{r-1}(M))) &= \mu^{s-2}C_{s-1}(h(\mu^{r-2}C_{r-1}(M))) \\ &= \mu^{s-2}C_{s-1}(\mu^{s-2}C_{s-1}(h(M))) \\ &= \mu^{s-2}(\mu^{s-2})^{s-1}C_{s-1}(C_{s-1}(h(M))). \end{aligned}$$

Since $R(h(M)) < s$, by Lemma 1.1(l), (m), it follows that $R(C_{s-1}(h(M)) \leqslant 1$ and so,

$$R(C_{s-1}(C_{s-1}(h(M)))) = 0.$$

This yields that,

$$h(\mu^{r-2}C_{r-1}(\mu^{r-2}C_{r-1}(M))) = \mu^{s-2}(\mu^{s-2})^{s-1}0_s = 0_s = h(0_r).$$

Since $\ker(h) = \{0_r\}$, it follows that,

$$\begin{aligned} &\mu^{r-2}C_{r-1}(\mu^{r-2}C_{r-1}(M)) = 0_r \\ \Rightarrow\ &\mu^{r-2}(\mu^{r-2})^{r-1}C_{r-1}(C_{r-1}(M)) = 0_r \\ \Rightarrow\ &C_{r-1}(C_{r-1}(M)) = 0_r. \end{aligned}$$

This shows that $R(C_{r-1}(C_{r-1}(M))) = 0$. Therefore, by Lemma 1.1(k)-(m), it follows that,

$$R(C_{r-1}(M)) \leqslant r - 2,$$

and hence, $R(M) \leqslant r - 1$. This contradicts with the fact that $R(M) = r$. So, we conclude that $R(h(M)) = s$.

(c) $\Rightarrow$ (d): By Lemma 1.1(k) and Lemma 2.3(c), we obtain,

$$\begin{aligned} R(M + \gamma N) = r &\Leftrightarrow R(h(M + \gamma N)) = s \\ &\Leftrightarrow R(C_{s-1}(h(M + \gamma N)) = s \\ &\Leftrightarrow R(C_{s-1}(h(M) + \gamma h(N)) = s \\ &\Leftrightarrow R(h(M) + \gamma h(N)) = s. \end{aligned}$$

(d) $\Rightarrow$ (a): Let $X, Y \in \mathcal{H}_r(\mathbb{F})$ with $h(X) = h(Y)$. Hence, by Lemma 2.5(a), $R(X - Y + W) = r$ for some $W \in \mathcal{H}_r(\mathbb{F})$ with $R(W) = r - R(X - Y)$. On the other hand, we have,

$$C_{s-1}(h(W)) = C_{s-1}(h(W - Y + Y)).$$

By Lemma 2.3(c), it follows that,

$$\begin{aligned} C_{s-1}(h(W)) &= C_{s-1}(h(W-Y)+h(Y)) \\ &= C_{s-1}(h(W-Y)+h(X)) \\ &= C_{s-1}(h(W-Y+X)) \\ &= C_{s-1}(h(X-Y+W)). \end{aligned}$$

Since $R(X-Y+W) = r$, we have $R(h(X-Y+W)) = s$. So, by Lemma 1.1(k), we obtain $R(C_{s-1}(h(X-Y+W))) = s$, which gives that $R(C_{s-1}(h(W)) = s$ and so, $R(h(W)) = s$. This implies that $R(W) = r$. Hence, $R(X-Y) = 0$, which gives that $X - Y = 0_r$. Consequently, $X = Y$, which implies that h is injective. □

In the following, we demonstrate that h possesses nicer properties when it is injective.

Lemma 2.8. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let $||\mathbb{F}^-|| = 2$ or $||\mathbb{F}^-|| \geqslant r+2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. If h is injective, then $h(M+\gamma N) = h(M) + \gamma h(N)$ for any $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$.*

Proof. To prove this, we break our proof into the following seven claims.

Claim 2.1. *If $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$ with $R(M+\gamma N) = r$, then,*

$$\frac{h(M+\gamma N)}{|h(M+\gamma N)|} = \frac{h(M)+\gamma h(N)}{|h(M)+\gamma h(N)|}, \tag{1}$$

and

$$\frac{h(M+\gamma N)}{|h(M+\gamma N)|} = \frac{h(M)+h(\gamma N)}{|h(M)+h(\gamma N)|}, \tag{2}$$

where $|h(M+\gamma N)|, |h(M)+\gamma h(N)|$ and $|h(M)+h(\gamma N)|$ are non-zero elements in $\mathbb{F}^-$.

In view of Lemma 2.3(c), we have,

$$C_{s-1}(h(M+\gamma N)) = C_{s-1}(h(M)+\gamma h(N)).$$

By Lemma 1.1(n), it gives that,

$$Z_s \mathrm{adj}(h(M+\gamma N))^{\sim} Z_s = Z_s \mathrm{adj}(h(M)+\gamma h(N))^{\sim} Z_s.$$

Since $|Z_s| \neq 0$, it follows that,

$$\mathrm{adj}(h(M+\gamma N))^{\sim} = \mathrm{adj}(h(M)+\gamma h(N))^{\sim},$$

and hence,

$$\mathrm{adj}(h(M+\gamma N)) = \mathrm{adj}(h(M)+\gamma h(N)).$$

From Lemma 2.7(d), $R(h(M+\gamma N)) = R(h(M)+\gamma h(N)) = s$. This shows that $h(M+\gamma N)$ and $h(M)+\gamma h(N)$ are invertible. By the fact of adjoint matrix, we get,

$$|h(M+\gamma N)|(h(M+\gamma N))^{-1} = |h(M)+\gamma h(N)|(h(M)+\gamma h(N))^{-1},$$

where $|h(M+\gamma N)|$ and $|h(M)+\gamma h(N)|$ are non-zero elements in $\mathbb{F}^-$. This yields that,

$$\frac{h(M+\gamma N)}{|h(M+\gamma N)|} = \frac{h(M)+\gamma h(N)}{|h(M)+\gamma h(N)|}.$$

On the other hand, by Lemma 2.3(c), we have,

$$C_{s-1}(h(M+\gamma N)) = C_{s-1}(h(M)+h(\gamma N)).$$

Hence, by using similar arguments as in above, we will obtain,

$$\frac{h(M+\gamma N)}{|h(M+\gamma N)|} = \frac{h(M)+h(\gamma N)}{|h(M)+h(\gamma N)|},$$

where $|h(M+\gamma N)|$ and $|h(M)+h(\gamma N)|$ are non-zero elements in $\mathbb{F}^-$.

Claim 2.2. *$h(\gamma M) = \gamma h(M)$ for any $\gamma \in \mathbb{F}^-$ and $M \in \mathcal{H}_r(\mathbb{F})$ with $R(M) = r$.*

Clearly, the claim holds when $\gamma = 0$. Thus, we consider for $\gamma \neq 0$. Since $R(\gamma M) = R(M) = r$, by Lemma 2.5(c), we have $R(N+\gamma M) = r$ for some $N \in \mathcal{H}_r(\mathbb{F})$ with $0 < R(N) < r$. From (1) and (2), we obtain,

$$\frac{h(N)+\gamma h(M)}{|h(N)+\gamma h(M)|} = \frac{h(N)+h(\gamma M)}{|h(N)+h(\gamma M)|}.$$

For the purpose of convenience, we let $\tau_1 = |h(N)+\gamma h(M)|$ and $\tau_2 = |h(N)+h(\gamma M)|$. It follows that,

$$\frac{h(N)+\gamma h(M)}{\tau_1} = \frac{h(N)+h(\gamma M)}{\tau_2}, \tag{3}$$

which yields that,

$$(\tau_2-\tau_1)h(N) = \tau_1 h(\gamma M) - \tau_2\gamma h(M).$$

From (1), we have,

$$h(\gamma M) = \kappa h(M),$$

where $\kappa = \dfrac{\gamma|h(\gamma M)|}{|\gamma h(M)|} \neq 0$. This gives that,

$$(\tau_2-\tau_1)h(N) = (\tau_1\kappa - \tau_2\gamma)h(M).$$

In view of Lemma 2.7(c), it follows that $R(h(M)) = s$ and $R(h(N)) \neq s$. This shows that $h(M)$ and $h(N)$ are linearly independent. So, we are forced to conclude that $\tau_2-\tau_1 = 0$, which implies that $\tau_1 = \tau_2$. Consequently, from (3), we obtain,

$$h(\gamma M) = \gamma h(M).$$

Claim 2.3. *For any $M, N \in \mathcal{H}_r(\mathbb{F})$ with $R(M+N) = r$, if $R(M) \neq r$ and $R(N) = r$, then $h(M+N) = h(M)+h(N)$.*

Obviously, the claim holds when $M = 0_r$. Hence, we focus on the case where $M \neq 0_r$. We consider the case where $||\mathbb{F}^-|| \geqslant r+2$. From Lemma 2.5(e), we have $R(M+N+\kappa N) = r$ for some $\kappa \in \mathbb{F}^-$ with $\kappa \neq 0$. Therefore, by (1), we have,

$$\frac{h(M+N+\kappa N)}{|h(M+N+\kappa N)|} = \frac{h(M+N)+\kappa h(N)}{|h(M+N)+\kappa h(N)|}.$$

On the other hand, by (1), we have,

$$\begin{aligned}\frac{h(M+N+\kappa N)}{|h(M+N+\kappa N)|} &= \frac{h(M+(\kappa+1)N)}{|h(M+(\kappa+1)N)|}\\ &= \frac{h(M)+(\kappa+1)h(N)}{|h(M)+(\kappa+1)h(N)|}\\ &= \frac{h(M)+h(N)+\kappa h(N)}{|h(M)+h(N)+\kappa h(N)|}.\end{aligned}$$

This yields that,

$$\frac{h(M+N)+\kappa h(N)}{|h(M+N)+\kappa h(N)|} = \frac{h(M)+h(N)+\kappa h(N)}{|h(M)+h(N)+\kappa h(N)|}.$$

Let $\tau_1 = \dfrac{|h(M+N)+\kappa h(N)|}{|h(M)+h(N)+\kappa h(N)|}$. Hence,

$$h(M+N)+\kappa h(N) = \tau_1(h(M)+h(N)+\kappa h(N)). \tag{4}$$

From (1), we see that,

$$h(M+N) = \tau_2(h(M)+h(N)),$$

where $\tau_2 = \dfrac{|h(M+N)|}{|h(M)+h(N)|}$. So, we have,

$$\begin{aligned}&\tau_2(h(M)+h(N))+\kappa h(N) = \tau_1(h(M)+h(N)+\kappa h(N))\\ \Rightarrow\ &(\tau_2-\tau_1)h(M)+(\tau_2-\tau_1+\kappa-\tau_1\kappa)h(N) = 0.\end{aligned}$$

By Lemma 2.7(c), we see that $R(h(M)) \neq s$ and $R(h(N)) = s$. This means that $h(M)$ and $h(N)$ are linearly independent. So, we are forced to conclude that $\tau_2 - \tau_1 = 0$ and $\tau_2 - \tau_1 + \kappa - \tau_1\kappa = 0$. This leads to $\kappa(1-\tau_1) = 0$. Since $\kappa \neq 0$, it follows that $1-\tau_1 = 0$, which gives that $\tau_1 = 1$. Consequently, we obtain,

$$h(M+N) = h(M)+h(N),$$

by (4). Next, we consider the case where $||\mathbb{F}^-|| = 2$. Since $R(M+N) = r$, from (1), we have,

$$\frac{h(M+N)}{|h(M+N)|} = \frac{h(M)+h(N)}{|h(M)+h(N)|}.$$

Since $||\mathbb{F}^-|| = 2$, it follows that $\mathbb{F}^- = \{0,1\}$. So, we conclude that $|h(M+N)| = |h(M)+h(N)| = 1$. Consequently, we obtain $h(M+N) = h(M)+h(N)$.

Claim 2.4. *$h(\gamma M) = \gamma h(M)$ for any $\gamma \in \mathbb{F}^-$ and $M \in \mathcal{H}_r(\mathbb{F})$ with $R(M) \neq r$.*

The result clearly true for $\gamma = 0$ or $M = 0_r$. Hence, we consider the case where $\gamma \neq 0$ and $M \neq 0_r$. Since $R(\gamma M) = R(M) \neq r$, we have $R(\gamma M + N) = r$ for some $N \in \mathcal{H}_r(\mathbb{F})$ with $R(N) = r$ by Lemma 2.5(c). It follows that $R(M+\gamma^{-1}N) = R(\gamma M + N) = r$. By Claim 2.3, we have $h(\gamma M + N) = h(\gamma M) + h(N)$. On the other hand, in view of Claims 2.2 and 2.3, we have,

$$\begin{aligned}h(\gamma M+N) &= h(\gamma(M+\gamma^{-1}N)) = \gamma h(M+\gamma^{-1}N) = \gamma(h(M)+h(\gamma^{-1}N))\\ &= \gamma h(M)+\gamma h(\gamma^{-1}N) = \gamma h(M)+h(N).\end{aligned}$$

Consequently, we obtain $h(\gamma M) = \gamma h(M)$.

Claim 2.5. *For any $M, N \in \mathcal{H}_r(\mathbb{F})$ with $R(M+N) = r$, if M and N are linearly independent, then $h(M)$ and $h(N)$ are linearly independent.*

Assume that $h(M)$ and $h(N)$ are linearly dependent for some $M, N \in \mathcal{H}_r(\mathbb{F})$ with M, N are linearly independent. Thus, $h(M) = ch(N)$ for some $c \in \mathbb{F}^-$. By Lemma 2.7(d), we obtain $R(h(M) + h(N)) = s$. This leads to $R((1+c)h(N)) = s$ and so, $R(h(N)) = R((1+c)h(N)) = s$. Hence, from Lemma 2.7(c), we have $R(N) = s$. Therefore, we have $h(M) = ch(N) = h(cN)$ by Claim 2.2. Since h is injective, it follows that $M = cN$. This shows that M and N are linearly dependent, which is a contradiction.

Claim 2.6. $h(M+N) = h(M) + h(N)$ *for any* $M, N \in \mathcal{H}_r(\mathbb{F})$ *with* $R(M+N) = r$.

Suppose that $||\mathbb{F}^-|| = 2$. By (1), we obtain,

$$\frac{h(M+N)}{|h(M+N)|} = \frac{h(M)+h(N)}{|h(M)+h(N)|}.$$

Since $||\mathbb{F}^-|| = 2$, it follows that $\mathbb{F}^- = \{0, 1\}$. So, we conclude that $|h(M+N)| = |h(M)+h(N)| = 1$. Consequently, we obtain $h(M+N) = h(M)+h(N)$. Here, we consider for $||\mathbb{F}^-|| \geqslant r+2$. Assume that M and N are linearly independent. From Lemma 2.5(e), we have $R(M+N+\kappa N) = r$ for some $\kappa \in \mathbb{F}^-$ with $\kappa \neq 0$. Therefore, by using similar arguments as in the proof of Claim 2.3, we get,

$$h(M+N) + \kappa h(N) = \tau_1(h(M) + h(N) + \kappa h(N)), \tag{5}$$

and

$$(\tau_2 - \tau_1)h(M) + (\tau_2 - \tau_1 + \kappa - \tau_1\kappa)h(N) = 0,$$

where $\tau_1 = \dfrac{|h(M+N)+\kappa h(N)|}{|h(M)+h(N)+\kappa h(N)|}$ and $\tau_2 = \dfrac{|h(M+N)|}{|h(M)+h(N)|}$. Since M and N are linearly independent, by Claim 2.5, $h(M)$ and $h(N)$ are also independent. So, we are forced to conclude that $\tau_2 - \tau_1 = 0$ and $\tau_2 - \tau_1 + \kappa - \tau_1\kappa = 0$. This leads to $\kappa(1-\tau_1) = 0$. Since $\kappa \neq 0$, it follows that $1 - \tau_1 = 0$, which gives that $\tau_1 = 1$. Consequently, we obtain,

$$h(M+N) = h(M) + h(N),$$

by (5). We next suppose that M and N are linearly dependent, then $N = cM$ for some $c \in \mathbb{F}^-$. By Claim 2.2, we get,

$$\begin{aligned} h(M+N) &= h(M+cM) = h((1+c)M) = (1+c)h(M) = h(M) + ch(M) \\ &= h(M) + h(cM) = h(M) + h(N). \end{aligned}$$

Claim 2.7. $h(M+N) = h(M) + h(N)$ *for any* $M, N \in \mathcal{H}_r(\mathbb{F})$.

In view of Lemma 2.5(b), $R(M+W) = R(M+N+W) = r$ for some $W \in \mathcal{H}_r(\mathbb{F})$. So, by Claim 2.6, we obtain,

$$\begin{aligned} & h(M+N+W) = h(M+W+N) \\ \Rightarrow\ & h(M+N) + h(W) = h(M+W) + h(N) \\ \Rightarrow\ & h(M+N) + h(W) = h(M) + h(W) + h(N) \\ \Rightarrow\ & h(M+N) = h(M) + h(N). \end{aligned}$$

Consequently, by combining all the cases above, we complete the proof. □

3 Characterization of Compound-Commuting Mappings on Skew-Hermitian Matrices

In the section, we determine the structures of $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ that satisfies [C1]. We start by proving $r = s$.

Lemma 3.1. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let $||\mathbb{F}^-|| = 2$ or $||\mathbb{F}^-|| \geqslant r+2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. If h is injective, then $r = s$.*

Proof. By Lemma 2.8, we have h is additive. Since $R(I_r) = r$, by Lemma 2.7(c), $R(h(I_r)) = s$. This gives that,

$$s = R(h(I_r)) = R(h(E_{11} + E_{22} + \ldots + E_{rr})).$$

Hence,

$$\begin{aligned} s &= R(h(E_{11}) + h(E_{22}) + \ldots + h(E_{rr})) \\ &\leqslant R(h(E_{11})) + R(h(E_{22})) + \ldots + R(h(E_{rr})). \end{aligned}$$

For any $i \in \{1, 2, \ldots, r\}$, since $R(E_{ii}) = 1$, hence we obtain $R(h(E_{ii})) \leqslant 1$ by Lemma 2.6(a). This gives that $s \leqslant r$. We assume that $s < r$. Therefore, in view of [10, Lemma 2.1], we have $R\left(\sum_{j\in J} h(E_{jj})\right) = s$ for some $J \subset \{1, 2, \ldots, r\}$. From Lemma 1.1(k), it follows that,

$$R\left(\mu^{s-2} C_{s-1}\left(\sum_{j\in J} h(E_{jj})\right)\right) = R\left(C_{s-1}\left(\sum_{j\in J} h(E_{jj})\right)\right) = s.$$

This leads to,

$$\begin{aligned} s &= R\left(\mu^{s-2} C_{s-1}\left(\sum_{j\in J} h(E_{jj})\right)\right) \\ &= R\left(\mu^{s-2} C_{s-1}\left(h\left(\sum_{j\in J} E_{jj}\right)\right)\right) \\ &= R\left(h\left(\mu^{r-2} C_{r-1}\left(\sum_{j\in J} E_{jj}\right)\right)\right). \end{aligned}$$

Since $||J|| < r$, we have $R\left(\sum_{j\in J} E_{jj}\right) = ||J|| < r$. Therefore, by Lemma 1.1(l), (m), it follows that $R\left(\mu^{r-2} C_{r-1}\left(\sum_{j\in J} E_{jj}\right)\right) = R\left(C_{r-1}\left(\sum_{j\in J} E_{jj}\right)\right) \leqslant 1$. So, by Lemma 2.6(a), we obtain $s = R\left(h\left(\mu^{r-2} C_{r-1}\left(\sum_{j\in J} E_{jj}\right)\right)\right) \leqslant 1$, which is a contradiction. Consequently, we obtain $r = s$. □

Next, we determine the form of h.

Lemma 3.2. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let $||\mathbb{F}^-|| = 2$ or $||\mathbb{F}^-|| \geqslant r+2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. If h is injective, then there exist some $\eta \in \mathbb{F}^-$ with $\eta \neq 0$, $G \in \mathbb{M}_r(\mathbb{F})$ with $R(G) = r$, and $\rho\colon \mathbb{F} \to \mathbb{F}$ with ρ is a non-zero field homomorphism of $\mathbb{F}$ satisfying $\rho(\overline{x}) = \overline{\rho(x)}$ for any $x \in \mathbb{F}$ such that,*

$$h(M) = \eta G M^\rho \overline{G}^T \text{ for any } M \in \mathcal{H}_r(\mathbb{F}).$$

Proof. By Lemma 2.6(a), we have $R(h(M)) \leqslant 1$ for any $M \in \mathcal{H}_r(\mathbb{F})$ with $R(M) = 1$. By Lemma 2.8, we have h is additive. Hence, h is a rank-one non-increasing additive mapping. On the other hand, since $R(I_r) = r$, by Lemma 2.7(c), we obtain $R(h(I_r)) = s$. It follows that h has an image with rank s. Therefore, in view of [17, Main Theorem, p. 603] and [18, Theorem 2.1 and Remark 2.4], we obtain,

$$h(M) = \eta G M^\rho \overline{G}^T \text{ for any } M \in \mathcal{H}_r(\mathbb{F}),$$

where $\eta \in \mathbb{F}^-$, $G \in \mathbb{M}_r(\mathbb{F})$, and $\rho\colon \mathbb{F} \to \mathbb{F}$ with ρ is a field homomorphism of $\mathbb{F}$ satisfying $\rho(\overline{x}) = \overline{\rho(x)}$ for any $x \in \mathbb{F}$. If $\eta = 0$ or ρ is a zero field homomorphism of $\mathbb{F}$, then $h(M) = \eta G M^\rho \overline{G}^T = 0_s$ for any $M \in \mathcal{H}_r(\mathbb{F})$, which implies that $R(h(M)) = 0$ for any $M \in \mathcal{H}_r(\mathbb{F})$. Besides that, if $R(G) \neq r$, then $||h(M)|| = \left|\left|\eta G M^\rho \overline{G}^T\right|\right| = 0$ for any $M \in \mathcal{H}_r(\mathbb{F})$, which implies that $R(h(M)) \neq s$. These three cases show that there does not exist any image of h with rank s, which is a contradiction. Consequently, $\eta \neq 0$, $R(G) = r$, and ρ is a non-zero field homomorphism of $\mathbb{F}$. $\square$

Subsequently, we investigate the structures of h.

Lemma 3.3. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let $||\mathbb{F}^-|| = 2$ or $||\mathbb{F}^-|| \geqslant r+2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. If h is injective, and,*

$$h(M) = \eta G M^\rho \overline{G}^T \text{ for any } M \in \mathcal{H}_r(\mathbb{F}),$$

where $\eta \in \mathbb{F}^-$ with $\eta \neq 0$, $G \in \mathbb{M}_r(\mathbb{F})$ with $R(G) = r$, and $\rho\colon \mathbb{F} \to \mathbb{F}$ with ρ is a non-zero field homomorphism of $\mathbb{F}$ satisfying $\rho(\overline{x}) = \overline{\rho(x)}$ for any $x \in \mathbb{F}$, then ρ is the identity mapping or $\rho = -$.

Proof. In view of Lemma 2.8, we have $h(yI_r) = yh(I_r)$ for any $y \in \mathbb{F}^-$. It follows that for any $y \in \mathbb{F}^-$,

$$\begin{aligned} h(yI_r) &= yh(I_r) \\ \Rightarrow \eta G (yI_r)^\rho \overline{G}^T &= y\eta G I_r^\rho \overline{G}^T \\ \Rightarrow (yI_r)^\rho &= yI_r^\rho \\ \Rightarrow \rho(y) I_r &= yI_r \\ \Rightarrow \rho(y) &= y. \end{aligned}$$

Let $y \in \mathbb{F}$ with $y \neq 0$. Note that, $y + \overline{y}, y\overline{y} \in \mathbb{F}^-$. It follows that,

$$\begin{aligned} \rho(y+\overline{y}) &= y+\overline{y} \quad \text{and} \quad \rho(y\overline{y}) = y\overline{y} \\ \Rightarrow \rho(y)+\rho(\overline{y}) &= y+\overline{y} \quad \text{and} \quad \rho(y)\rho(\overline{y}) = y\overline{y} \\ \Rightarrow \rho(y)(y+\overline{y}-\rho(y)) &= y\overline{y} \\ \Rightarrow \rho(y)^2 - (y+\overline{y})\rho(y) + y\overline{y} &= 0 \\ \Rightarrow (\rho(y)-y)(\rho(y)-\overline{y}) &= 0 \\ \Rightarrow \rho(y) &= y \quad \text{or} \quad \rho(y) = \overline{y}. \end{aligned}$$

This shows that ρ is the identity mapping or $\rho = -$. □

Lemma 3.4. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let $||\mathbb{F}^-|| = 2$ or $||\mathbb{F}^-|| \geqslant r+2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfying [C1]. If h is injective, and,*

$$h(M) = \eta G M^{\rho} \overline{G}^T \text{ for any } M \in \mathcal{H}_r(\mathbb{F}),$$

where $\eta \in \mathbb{F}^-$ with $\eta \neq 0$, $G \in \mathbb{M}_r(\mathbb{F})$ with $R(G) = r$, and $\rho\colon \mathbb{F} \to \mathbb{F}$ with ρ is ρ is the identity mapping or $\rho = -$, then $C_{r-1}(G) = gG$ for some non-zero element $g \in \mathbb{F}$ with $\eta^{r-2} g\overline{g} = 1$.

Proof. In view of Lemma 3.1, we obtain $r = s$. Let $M \in \mathbb{M}_r(\mathbb{F})$. Hence, we have,

$$\begin{aligned} h(\mu^{r-2}C_{r-1}(M)) &= \mu^{r-2}C_{r-1}(h(M)) \\ \Rightarrow \eta G(\mu^{r-2}C_{r-1}(M))^{\rho}\overline{G}^T &= \mu^{r-2}C_{r-1}(\eta G M^{\rho}\overline{G}^T) \\ \Rightarrow \eta\rho(\mu^{r-2})GC_{r-1}(M)^{\rho}\overline{G}^T &= \mu^{r-2}\eta^{r-1}C_{r-1}(G)C_{r-1}(M)^{\rho}\overline{C_{r-1}(G)}^T \\ \Rightarrow \eta\mu^{r-2}GC_{r-1}(M)^{\rho}\overline{G}^T &= \mu^{r-2}\eta^{r-1}C_{r-1}(G)C_{r-1}(M)^{\rho}\overline{C_{r-1}(G)}^T. \end{aligned}$$

It follows that,

$$\begin{aligned} GC_{r-1}(M)^{\rho}\overline{G}^T &= \eta^{r-2}C_{r-1}(G)C_{r-1}(M)^{\rho}\overline{C_{r-1}(G)}^T \\ \Rightarrow C_{r-1}(G)^{-1}GC_{r-1}(M)^{\rho} &= \eta^{r-2}C_{r-1}(M)^{\rho}\overline{C_{r-1}(G)}^T(\overline{G}^T)^{-1} \\ \Rightarrow C_{r-1}(G)^{-1}GC_{r-1}(M)^{\rho} &= \eta^{r-2}C_{r-1}(M)^{\rho}((\overline{C_{r-1}(G)^{-1}G})^T)^{-1}. \end{aligned}$$

Let $U = C_{r-1}(G)^{-1}G$ and $V = \eta^{r-2}((\overline{C_{r-1}(G)^{-1}G})^T)^{-1}$. We set $M = I_r$. Since $C_{r-1}(I_r)^{\rho} = I_r^{\rho}$ and $I_r^{\rho} = I_r$, we obtain,

$$UI_r = I_rV \Rightarrow U = V.$$

Next, we set $M = I_r - E_{r+1-i,r+1-i}$, where $i \in \mathbb{N}$ with $1 \leqslant i \leqslant r$. By Lemma 1.1(o), we know that $C_{r-1}(M)^{\rho} = E_{ii}^{\rho} = E_{ii}$. It follows that $UE_{ii} = E_{ii}U$. For any $j \in \mathbb{N}$ with $1 \leqslant j \neq i \leqslant r$, we have,

$$\begin{aligned} (UE_{ii})_{ij} &= (E_{ii}U)_{ij} \\ \Rightarrow \sum_{k=1}^{r} U_{ik}(E_{ii})_{kj} &= \sum_{k=1}^{r}(E_{ii})_{ik}U_{kj} \\ \Rightarrow \sum_{k=1}^{r} U_{ik}(0) &= (E_{ii})_{ii}U_{ij} \\ \Rightarrow 0 &= (1)U_{ij} \\ \Rightarrow U_{ij} &= 0. \end{aligned}$$

This shows that U is a diagonal matrix. Subsequently, we set,

$$M = I_r - E_{r+1-i,r+1-i} - E_{r+1-j,r+1-j} + (-1)^{i+j+1} E_{r+1-j,r+1-i},$$

where $i, j \in \mathbb{N}$ with $1 \leqslant i \neq j \leqslant r$. By Lemma 1.1(p), we know that $C_{r-1}(M)^\rho = E_{ij}^\rho = E_{ij}$. This implies that,

$$\begin{aligned} UE_{ij} &= E_{ij}U \\ \Rightarrow (UE_{ij})_{ij} &= (E_{ij}U)_{ij} \\ \Rightarrow \sum_{k=1}^{n} U_{ik}(E_{ij})_{kj} &= \sum_{k=1}^{n} (E_{ij})_{ik} U_{kj} \\ \Rightarrow U_{ii}(E_{ij})_{ij} &= (E_{ij})_{ij} U_{jj} \\ \Rightarrow U_{ii}(1) &= (1)U_{jj} \\ \Rightarrow U_{ii} &= U_{jj}. \end{aligned}$$

This shows that U is a diagonal matrix, where each diagonal entry is equal. Hence, $U = uI_r$ for some $u \in \mathbb{F}$ and so, $U = g^{-1}I_r$ for some $g \in \mathbb{F}$. This gives that,

$$\begin{aligned} C_{r-1}(G)^{-1}G &= g^{-1}I_r \quad \text{and} \quad \eta^{r-2}((\overline{C_{r-1}(G)^{-1}G})^T)^{-1} = g^{-1}I_r \\ \Rightarrow C_{r-1}(G) &= gG \qquad \text{and} \qquad\qquad C_{r-1}(G) = \overline{g}^{-1}\eta^{2-r}G \\ \Rightarrow g &= \overline{g}^{-1}\eta^{2-r} \\ \Rightarrow \eta^{r-2} g\overline{g} &= 1. \end{aligned}$$

Consequently, we obtain $C_{r-1}(G) = gG$ and $\eta^{r-2}g\overline{g} = 1$, as desired. □

We characterize h in the next proposition.

Proposition 3.1. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let $||\mathbb{F}^-|| = 2$ or $||\mathbb{F}^-|| \geqslant r + 2$. Let μ be a fixed arbitrary non-zero element in $\mathbb{F}^- \cup S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be an injective mapping. Then,*

$$h(\mu^{r-2}C_{r-1}(M + \gamma N)) = \mu^{s-2}C_{s-1}(h(M) + \gamma h(N)),$$

for any $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$ if and only if $r = s$, and there exist some non-zero elements $\eta \in \mathbb{F}^-$, $g \in \mathbb{F}$ and invertible matrix $G \in \mathbb{M}_r(\mathbb{F})$ with $\eta^{r-2}g\overline{g} = 1$ and $C_{r-1}(G) = gG$ such that $h(M) = \eta GM\overline{G}^T$ for any $M \in \mathcal{H}_r(\mathbb{F})$ or $h(M) = \eta G\bar{M}\overline{G}^T$ for any $M \in \mathcal{H}_r(\mathbb{F})$.

Proof. In view of Lemmas 3.1–3.4, the necessity part is obtained immediately. Conversely, we let $\rho\colon \mathbb{F} \to \mathbb{F}$ be a mapping with ρ is the identity mapping or $\rho = -$. Hence, we have $h(M) = \eta GM^\rho\overline{G}^T$

for any $M \in \mathcal{H}_r(\mathbb{F})$. Therefore, for any $M, N \in \mathbb{F}$ and $\gamma \in \mathbb{F}^-$, we obtain,

$$\begin{aligned}
h(\mu^{r-2}C_{r-1}(M+\gamma N)) &= \eta G(\mu^{r-2}C_{r-1}(M+\gamma N))^{\rho}\overline{G}^T \\
&= \rho(\mu^{r-2})\eta GC_{r-1}(M+\gamma N)^{\rho}\overline{G}^T \\
&= \mu^{r-2}\eta GC_{r-1}(M+\gamma N)^{\rho}\overline{G}^T \\
&= \mu^{r-2}\eta(g^{-1}C_{r-1}(G))C_{r-1}(M+\gamma N)^{\rho}(\overline{g}^{-1}C_{r-1}(\overline{G}^T)) \\
&= \mu^{r-2}\eta g^{-1}\overline{g}^{-1}C_{r-1}(G(M+\gamma N)^{\rho}\overline{G}^T) \\
&= \mu^{r-2}\eta\eta^{r-2}C_{r-1}(G(M^{\rho}+\rho(\gamma)N^{\rho})\overline{G}^T) \\
&= \mu^{r-2}\eta^{r-1}C_{r-1}(GM^{\rho}\overline{G}^T+\gamma GN^{\rho}\overline{G}^T) \\
&= \mu^{r-2}C_{r-1}(\eta GM^{\rho}\overline{G}^T+\gamma\eta GN^{\rho}\overline{G}^T) \\
&= \mu^{r-2}C_{r-1}(h(M)+\gamma h(N)) \\
&= \mu^{s-2}C_{s-1}(h(M)+\gamma h(N)).
\end{aligned}$$
□

Finally, by applying Proposition 3.1 and Lemma 2.2, we obtain a general form of the compound-commuting mapping on skew-Hermitian matrices that satisfies [S1], as follows:

Theorem 3.1. *Let $\mathbb{F}$ be a field with involution $-$ and let r, s be even integers with $r, s > 2$. Let $||\mathbb{F}^-|| = 2$ or $||\mathbb{F}^-|| \geqslant r+2$. Let $\zeta\colon \mathcal{SH}_r(\mathbb{F}) \to \mathcal{SH}_s(\mathbb{F})$ be an injective mapping. Then,*

$$\zeta(C_{r-1}(M+\gamma N)) = C_{s-1}(\zeta(M)+\gamma\zeta(N)),$$

for any $M, N \in \mathcal{SH}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$ if and only if $r = s$, and there exist some non-zero elements $\eta \in \mathbb{F}^-, g \in \mathbb{F}$ and invertible matrix $G \in \mathbb{M}_r(\mathbb{F})$ with $\eta^{r-2}g\overline{g} = 1$ and $C_{r-1}(G) = gG$ such that $\zeta(M) = \eta GM\overline{G}^T$ for any $M \in \mathcal{H}_r(\mathbb{F})$ or $\zeta(M) = \eta G\bar{M}\overline{G}^T$ for any $M \in \mathcal{H}_r(\mathbb{F})$.

Proof. The sufficiency part can be proved by using similar arguments as in the proof of Proposition 3.1. Now, we consider the necessity part. Let μ be a fixed arbitrary non-zero element in $S\mathbb{F}^-$ and let $h\colon \mathcal{H}_r(\mathbb{F}) \to \mathcal{H}_s(\mathbb{F})$ be a mapping satisfies $h(W) = \mu^{-1}\zeta(\mu W)$ for any $W \in \mathcal{H}_r(\mathbb{F})$. By Lemma 2.2, h satisfies,

$$h(\mu^{r-2}C_{r-1}(M+\gamma N)) = \mu^{s-2}C_{s-1}(h(M)+\gamma h(N)),$$

for any $M, N \in \mathcal{H}_r(\mathbb{F})$ and $\gamma \in \mathbb{F}^-$. Therefore, in view of Proposition 3.1, it follows that $r = s$, and there exist some non-zero elements $c \in \mathbb{F}^-, g \in \mathbb{F}$ and invertible matrix $G \in \mathbb{M}_r(\mathbb{F})$ with $c^{r-2}g\overline{g} = 1$ and $C_{r-1}(G) = gG$ such that $h(M) = cGM\overline{G}^T$ for any $M \in \mathcal{H}_r(\mathbb{F})$ or $h(M) = cG\bar{M}\overline{G}^T$ for any $M \in \mathcal{H}_r(\mathbb{F})$. Let $\rho\colon \mathbb{F} \to \mathbb{F}$ be a mapping with ρ is the identity mapping or $\rho = -$. Thus, we have $h(M) = cGM^{\rho}\overline{G}^T$ for any $M \in \mathcal{H}_r(\mathbb{F})$. For any $M \in \mathcal{SH}_r(\mathbb{F})$, we have,

$$\begin{aligned}
\zeta(M) &= \zeta(\mu(\mu^{-1}M)) \\
&= \mu h(\mu^{-1}M) \\
&= \mu cG(\mu^{-1}M)^{\rho}\overline{G}^T \\
&= \mu\rho(\mu^{-1})cGM^{\rho}\overline{G}^T \\
&= \mu(\pm\mu^{-1})cGM^{\rho}\overline{G}^T \\
&= \pm cGM^{\rho}\overline{G}^T.
\end{aligned}$$

Let $\eta = \pm c$. Hence, we have $\eta \in \mathbb{F}^-$ and $\eta^{r-2} g\overline{g} = 1$. Consequently, we obtain $\zeta(M) = \eta GM\overline{G}^T$ for any $M \in \mathcal{SH}_r(\mathbb{F})$ or $\zeta(M) = \eta G\bar{M}\overline{G}^T$ for any $M \in \mathcal{SH}_r(\mathbb{F})$. We are done. $\square$

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